Plasma Brake Approximate Trajectory
Part I: Geocentric Motion
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Introduction
Aim of the work

Motivation

- Space debris is a major concern, especially for LEO missions.
- IADC guidelines: residual orbit life $\leq 25$ years for out-of-order satellites.
- Plasma brake is a promising innovation in terms of deorbiting technologies.

Aim of the work

- Analytical approximation of the geocentric trajectory of a spacecraft equipped with a plasma brake
- Estimation of the decay time
Plasma brake concept

Plasma brake

- Derivation of the electric solar wind sail (E-sail) concept
- Application: deorbiting from LEO

Working principle

- A tether is released in the ionosphere and electrostatically charged by a voltage source.
- The collisions between ions and tether generate a Coulomb drag.
- The Coulomb drag lowers the spacecraft altitude.
Geometrical sketch of the problem

Note: subscript 0 identifies initial conditions.
Mathematical model
2D Equations of motion

- Polar reference frame
- Two-dimensional motion (thrust vector belongs to the orbital plane)
- Coulomb drag opposite to spacecraft velocity

\[
\begin{align*}
\ddot{r} - r\dot{\theta}^2 &= -\frac{\mu_\oplus}{r^2} - \epsilon \frac{\mu_\oplus}{r_0^2} \frac{\dot{r}}{\sqrt{\dot{r}^2 + (r\dot{\theta})^2}} \\
2r\ddot{\theta} + 2\dot{r}\dot{\theta} &= -\epsilon \frac{\mu_\oplus}{r_0^2} \frac{r\dot{\theta}}{\sqrt{\dot{r}^2 + (r\dot{\theta})^2}}
\end{align*}
\]

Nomenclature

- \( \mu_\oplus \triangledown \) Earth’s gravitational parameter
- \( D_c \triangledown \) Coulomb drag magnitude
- \( \epsilon \triangledown \) \( D_c / \sqrt{\mu_\oplus/r_0^2} \)

Problems

- Coulomb drag is very complex to model.
- The integration has a high computational cost.
Non-singular orbital elements

Introduction of **non-dimensional orbital parameters** (Bombardelli et al., 2011):

\[
q_1 \equiv \frac{e}{
\tilde{H}} \cos \omega, \quad q_2 \equiv \frac{e}{
\tilde{H}} \sin \omega, \quad q_3 \equiv \frac{1}{
\tilde{H}}
\]

where \( \tilde{H} \) is the dimensionless angular momentum magnitude.

**Note:** classical orbital parameters can be recovered as

\[
a = \frac{r_0}{q_3 - q_1^2 - q_2^2}
\]

\[
e = \sqrt{\frac{q_1^2 + q_2^2}{q_3}}
\]

\[
r = \frac{r_0}{q_3 + q_1 q_3 \cos \theta + q_2 q_3 \sin \theta}
\]

\[
\omega = \arctan \left( \frac{q_2}{q_1} \right)
\]

\[
v_r = \sqrt{\frac{\mu_{\odot}}{r_0}} \left( q_1 \sin \theta - q_2 \cos \theta \right)
\]

\[
v_\theta = \sqrt{\frac{\mu_{\odot}}{r_0}} \left( q_1 \cos \theta + q_2 \sin \theta + q_3 \right)
\]
Variational equations of modified parameters

The variational equations of modified parameters are (Coulomb drag opposite w.r.t. orbital velocity)

\[
\frac{d}{d\theta} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{\epsilon}{q_3 s^3 \sqrt{e^2 + 2e\cos\nu + 1}} \begin{bmatrix} s \sin\theta & (s + q_3) \cos\theta \\ -s \cos\theta & (s + q_3) \sin\theta \\ 0 & -q_3 \end{bmatrix} \begin{bmatrix} e \sin\nu \\ 1 + e \cos\nu \end{bmatrix}
\]

where

\[s = q_1 \cos\theta + q_2 \sin\theta + q_3, \quad e = \sqrt{\frac{q_1^2 + q_2^2}{q_3}}, \quad \nu = \theta - \arctan\left(\frac{q_2}{q_1}\right)\]

Initial conditions:

\[q_1 = \frac{e_0}{\tilde{H}_0}, \quad q_2 = 0, \quad q_3 = \frac{1}{\tilde{H}_0} \quad \text{with} \quad \tilde{H}_0 = \sqrt{1 + e_0 \cos\nu_0}\]
Introduction to expansion method

Fundamental hypothesis

The Coulomb drag magnitude is small w.r.t. the Earth’s gravity, i.e. $\epsilon \ll 1$

The dimensionless propulsive acceleration $\epsilon$ is a perturbation term in the asymptotic series expansion:

$$q_1 = q_{10} + \epsilon q_{11} + O(\epsilon^2)$$

$$q_2 = q_{20} + \epsilon q_{21} + O(\epsilon^2)$$

$$q_3 = q_{30} + \epsilon q_{31} + O(\epsilon^2)$$

where

- $q_{i0}$ are (constant) unperturbed terms
- $q_{i1}$ are first order perturbation terms
Evolution of modified parameters

Mathematical procedure

1. The series expansions are substituted in the variational equations.
2. Terms of the same perturbative order are equated.
3. Differential equations are integrated in the angular variable, maintaining constant the value of the perturbative parameter $\epsilon$.

Analytical expressions of modified parameters (circular initial orbit)

$$q_1 = -2\epsilon (\sin \theta - \sin \theta_0)$$
$$q_2 = 2\epsilon (\cos \theta - \cos \theta_0)$$
$$q_3 = \frac{1}{\tilde{H}_0} + \epsilon (\theta - \theta_0)$$

More complex expressions exist for the elliptic initial orbit case.
Approximate plasma brake model
Coulomb drag

Hypothesis
- Earth's magnetosphere effects are neglected (Janhunen, 2014).
- A constant drag per unit length is assumed.
- A Heytether is considered.

\[ D_c = 3.864 L_t m_i n v^2 \sqrt{\frac{\epsilon_0 V_a}{el n}} \exp\left(-\frac{m_i v^2}{2 el V_a}\right) \]

with
\[ V_a = \frac{2 |V_t|}{\ln\left(\frac{\epsilon_0 |V_t|}{el n_0 b_t r_w}\right)} \]

Nomenclature
- \( el \triangleq \) elementary charge
- \( m_i \triangleq \) ions mass
- \( n \triangleq \) plasma bulk density
- \( \epsilon_0 \triangleq \) vacuum permittivity
Approximate model (Orsini et al., 2017)

- Plasma density estimated with **geopotential model**
- Constant ions mass (atomic oxygen, \( m_i \approx 16 \text{ u} \)) and temperature
- Initial altitude taken as **reference altitude**

\[
D_c \approx D_{c0} \sqrt{\frac{n}{n_0}} \left(\frac{v}{v_0}\right)^2 f_1(v, v_0) f_2(n, n_0, v, v_0) f_3(n, n_0)
\]

\(D_{c0}\) is calculated with accurate ionosphere models (IRI).

**Approximation for LEOs (300 km \( \leq h \leq 1000 \text{ km}\) with \( \tilde{h} = 1000 \text{ km}\)**

- \(0.96 \leq f_1 \leq 1\)
- \((v/v_0)^2 \approx 1\)
- \(f_3 \approx 1\)
- \(1 \leq f_2 \leq 1.04\) \(\Rightarrow f_1 f_2 \approx 1\)

**Conservative assumption** with max error 10%

\[
D_c \approx D_{c0} \exp\left\{ -\frac{m_i \mu_\oplus}{4 k_B T} \left[\frac{h}{(R_\oplus + h)^2} - \frac{h_0}{(R_\oplus + h_0)^2}\right] \right\}
\]
Rectification procedure
Rectification procedure (1/2)

1. Select the **rectification coordinate** $\theta_r$ and calculate the corresponding values of the modified parameters $q_{ir}$.

2. Find the **altitude** $h_r$, the **eccentricity** $e_r$, the **true anomaly** $\nu_r$, the **apse line rotation angle** $\omega_r$, and the **dimensionless angular momentum** $\tilde{H}_r$ as functions of $q_{ir}$.

3. **Update** the magnitude of the Coulomb drag at the altitude $h_r$ by means of the approximate analytical model and calculate the new value of the perturbative parameter $\bar{\epsilon}$.

4. Define a new set of **auxiliary variables**

\[
\bar{\theta} \triangleq \theta - \omega, \quad \bar{\omega} \triangleq \omega - \omega_r, \quad \bar{q}_1 \triangleq \frac{e}{\tilde{H}} \cos \omega_r \quad \bar{q}_2 \triangleq \frac{e}{\tilde{H}} \sin \omega_r \quad \bar{q}_3 \triangleq \frac{1}{\tilde{H}}
\]
Rectification procedure (2/2)

5 The **initial conditions** for the new auxiliary variables are

\[
\bar{\theta}_0 = \nu_r, \quad \bar{\omega}_0 = 0, \quad \bar{q}_{10} = \frac{e_r}{\bar{H}_r}, \quad \bar{q}_{20} = 0, \quad \bar{q}_{30} = \frac{1}{\bar{H}_r}
\]

6 Calculate the **evolution of the new auxiliary variables** with the analytical expressions, substituting \(e_0, \bar{H}_0, \nu_0, \text{ and } \epsilon\) with \(e_r, \bar{H}_r, \nu_r, \bar{\epsilon}, \text{ and } \bar{\theta}\), respectively.

7 To get the **initial parameters**, apply the following **rotational matrix**

\[
\begin{bmatrix}
q_1(\theta) \\
q_2(\theta) \\
q_3(\theta)
\end{bmatrix} =
\begin{bmatrix}
\cos \omega_r & -\sin \omega_r & 0 \\
\sin \omega_r & \cos \omega_r & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\bar{q}_1(\bar{\theta}) \\
\bar{q}_2(\bar{\theta}) \\
\bar{q}_3(\bar{\theta})
\end{bmatrix}
\text{ for } \theta \geq \theta_r
\]
Case study
Deorbiting profile from $h_0 = 1000$ km, mean solar activity

Three different nanosatellites are analyzed in our simulations

|                  | $m$ [kg] | $L_t$ [m] | $|V_t|$ [V] |
|------------------|----------|-----------|------------|
| spacecraft 1     | 1.0      | 25        | 500        |
| spacecraft 2     | 4.0      | 100       | 1000       |
| spacecraft 3     | 10.0     | 300       | 1000       |

Comparison between the approximate method with 100 rectifications per year (red line) and an orbital propagator (black line)
Decay times from $h_0 = 1000 \, \text{km}$ to $300 \, \text{km}$

<table>
<thead>
<tr>
<th></th>
<th>Numerical</th>
<th>Analytical</th>
<th>Percentage error</th>
</tr>
</thead>
<tbody>
<tr>
<td>spacecraft 1</td>
<td>1317 days</td>
<td>1320 days</td>
<td>0.26%</td>
</tr>
<tr>
<td>spacecraft 2</td>
<td>924 days</td>
<td>928 days</td>
<td>0.38%</td>
</tr>
<tr>
<td>spacecraft 3</td>
<td>770 days</td>
<td>774 days</td>
<td>0.45%</td>
</tr>
</tbody>
</table>

Remarks

- The estimation of the geocentric trajectory is practically coincident with the output of an orbital propagator.
- The determination of the decay time is also very accurate.
- The approximated method requires a computational cost which is about 2 order of magnitudes smaller w.r.t. the numerical integration of the equations of motion.
- The estimated decay times are in accordance with IADC guidelines.
Conclusions
Conclusions

- **Approximate method** for the *geocentric trajectory* analysis of a spacecraft deorbiting from a LEO by means of a *plasma brake*

- Model based on the *small magnitude* of the Coulomb drag, which is expressed as a *function of the orbital altitude only*

- Results show very *small errors*, both in terms of geocentric trajectory and decay times, when compared to the outputs of an orbital propagator

- **Computational time** about *two order of magnitudes smaller* with respect to the numerical integration of the equations of motion, even with a high number of rectifications

- Decay times of a *plasma brake-enabled deorbiting* are *in accordance with international guidelines*
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