On the Beletskii equation

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Gravity gradient effect on attitude motion of a (dumbbell) spacecraft
Equilibrium configurations

Stable Equilibrium

Unstable Equilibrium
Formulas

Gravitational torque
wrt the c.o.m. G on a
particle \( dm \)

\[
d\vec{M} = \vec{\rho} \wedge d\vec{f}(\vec{r}) = \vec{\rho} \wedge \left(-\frac{\mu}{r^3} \vec{r}\right) dm
\]

\[
\vec{r} = \vec{r}_G + \vec{\rho}
\]

\[
d\vec{f}(\vec{r}) = -\frac{\mu}{r^3} \vec{r} = -\frac{\mu}{\|\vec{r} + \vec{r}_G\|^3} (\vec{r} + \vec{r}_G) \sim d\vec{f}(\vec{r}_G) + \nabla d\vec{f}(\vec{r}_G) \vec{\rho}
\]

\[
d\vec{f}(\vec{r}) \sim -\frac{\mu}{r_G^3} \vec{r}_G dm + \frac{\mu}{r_G^3} (3 \hat{r}_G \hat{r}_G - \mathbf{u}) \vec{\rho} dm
\]

Integrating over all the mass of the satellite

\[
\vec{M} = \int_B d\vec{M} = \int_B \vec{\rho} dm \wedge \left(-\frac{\mu}{r_G^3} \vec{r}_G\right) + \frac{\mu}{r_G^3} \int_B \vec{\rho} \wedge (3 \hat{r}_G \hat{r}_G - \mathbf{u}) \vec{\rho} dm
\]
Gravity Gradient Torque and Equations of attitude motion

\[ \vec{M} = 3 \frac{\mu}{r_G^3} \int_B \vec{\rho} \wedge \hat{r}_G \hat{r}_G \vec{\rho} \, dm = -3 \frac{\mu}{r_G^3} \int_B \hat{r}_G \wedge \vec{\rho} \vec{\rho} \hat{r}_G \, dm \]

Adding a «zero» we have

\[ \vec{M} = 3 \frac{\mu}{r_G^3} \vec{r}_G \wedge \int_B (\rho^2 \vec{u} - \vec{\rho} \vec{\rho}) \, dm \vec{r}_G \]

That is

\[ \vec{M} = 3 \frac{\mu}{r_G^3} \vec{r}_G \wedge \vec{I} \vec{r}_G \]

Gravitational Torque in body frame

\[ \vec{I} \vec{\dot{\Omega}} + \vec{\Omega} \wedge \vec{I} \vec{\dot{\Omega}} = 3 \frac{\mu}{r_G^3} \vec{r}_G \wedge \vec{I} \vec{r}_G \]
Body components and planar case

\[ \vec{r}_G = r_x \hat{x}_B + r_y \hat{y}_B + r_z \hat{z}_B \]

\[ \vec{\Omega} = \Omega_1 \hat{x}_B + \Omega_2 \hat{y}_B + \Omega_3 \hat{z}_B \]

**Planar Case** \( \Omega_1 = \Omega_2 = r_z = 0 \)

\[ \dot{\Omega}_3 = 3 \frac{\mu}{r^3} \frac{I_2 - I_1}{I_3} r_x r_y \]

\[ \dot{\Omega}_1 = -\frac{I_3 - I_2}{I_1} \Omega_3 \Omega_2 + 3 \frac{\mu}{r^3} \frac{I_3 - I_2}{I_1} r_y r_z \]

\[ \dot{\Omega}_2 = -\frac{I_1 - I_3}{I_2} \Omega_3 \Omega_1 + 3 \frac{\mu}{r^3} \frac{I_1 - I_3}{I_2} r_x r_z \]

\[ \dot{\Omega}_3 = -\frac{I_2 - I_1}{I_3} \Omega_2 \Omega_1 + 3 \frac{\mu}{r^3} \frac{I_2 - I_1}{I_3} r_x r_y \]

\[ \ddot{\psi} + \ddot{\theta} = -3 \frac{\mu}{r^3} \frac{I_2 - I_1}{I_3} \cos \psi \sin \psi \]

\[ \ddot{\psi} = -\frac{3}{2} \frac{\mu}{R^3} \frac{I_2 - I_1}{I_3} \sin 2\psi \]

\[ e = 0 \]
Consider elliptic orbits

\[ \ddot{\psi} + \dot{\theta} = -3 \frac{\mu}{r^3} \frac{I_2 - I_1}{I_3} \cos \psi \sin \psi \]

\[ (\ast)' = \frac{d \ast}{d \theta} \]

\[ r = \frac{\hbar^2/\mu}{1 + e \cos \theta} \]

\[ \dot{\theta} = \frac{\mu^2}{\hbar^3} (1 + e \cos \theta) \]

\[ \ddot{\theta} = 2 \left( \frac{\mu^2}{\hbar^3} \right)^2 (1 + e \cos \theta)^3 (-e \sin \theta) \]

\[ (1 + e \cos \theta) \phi'' - 2e \sin \theta \phi' + \alpha \sin \phi = 4e \sin \theta \]

\[ \phi = 2 \psi \]

\[ \alpha = 3 \frac{I_2 - I_1}{I_3} \in [0, 3] \]
In this paper the methods of Liapunov-Chetayev are applied to find conditions for the existence and stability of the relative equilibrium of a material body in orbit in a Newtonian central-force field and to consider the oscillations of the body around the position of relative equilibrium. The problem considered is an idealized motion, which actually occurs in the solar system (the motion of the moon relative to the earth and the possible motions of artificial earth satellites).
Periodic Solutions

A.Torzheskii: "Periodic solutions of the equation for two dimensional oscillations of a satellite in elliptic orbit", Cosmi Research, vol. 2, pp. 667-678,
Chaotic Dynamics


Generalized equations


Periodic solutions of small amplitude and small eccentricity: Beletskii transformation

Small amplitude

\[(1 + e \cos \theta)\phi'' - 2e \sin \theta \phi' + 3\alpha \phi = 4e \sin \theta\]

\[z = (1 + e \cos \theta)\frac{\phi}{2}\]

\[z'' + \frac{\alpha + e \cos \theta}{1 + e \cos \theta} z = 2e \sin \theta\]

Small eccentricity

\[z'' + \alpha z = 2e \sin \theta + (\alpha - 1)(e \cos \theta - e^2 \cos^2 \theta)z + \ldots\]

For \(e = 0\) the solution is:

\[
\begin{align*}
z &= D \cos \tau \\
D' &= 0 \\
\tau' &= \sqrt{\alpha}
\end{align*}
\]
Periodic solutions of small amplitude and small eccentricity: solution

For \( e \ll 1 \) search a solution:

\[
z = D \cos \tau + e u_1(D, \tau, \theta) + e^2 u_1(D, \tau, \theta) + \ldots
\]

with

\[
D' = e A_1(D) + e^2 A_2(D) + \ldots
\]

\[
\tau' = \sqrt{\alpha} + e B_1(D) + e^2 B_2(D) + \ldots
\]

Result

\[
z(\theta) = D \cos(\sqrt{\alpha} \theta + e^2 B_2 \theta + t \omega_0) + e u_1 + e^2 u_2
\]

\[
u_1 = \frac{2 \sin \theta}{\alpha - 1} + \frac{D(\alpha - 1)}{2} \cos((\Gamma - 1)\theta + \tau_0) - \frac{2 \sqrt{\alpha} - 1}{4\alpha - 1} \cos((\Gamma + 1)\theta + \tau_0)
\]

\[
u_2 = \frac{\sin(2\theta)}{\alpha - 4} + \frac{\alpha \sqrt{\alpha} + 1}{16} \frac{\sqrt{\alpha} + 1} {2\sqrt{\alpha} + 1} \cos((\Gamma - 2)\theta + \tau_0) + \frac{\sqrt{\alpha} - 1}{2\sqrt{\alpha} + 1} \sqrt{\alpha} + 2 \cos((\Gamma + 2)\theta + \tau_0)
\]

\[
\Gamma = \sqrt{\alpha} + e^2 B_2, \quad B_2 = \frac{3}{4} \frac{\sqrt{\alpha}(\alpha - 1)}{4\alpha - 1}
\]
Numerical solutions: resonances

Initial state = (0,0)

alpha = 1

Initial state = (0,0)

alpha = 1/4
Periodic Solutions by Newton-Raphson on the Poincarè map

\[ P : (\phi(\tau), \phi'(\tau)) \rightarrow (\phi(\tau + 2\pi), \phi'(\tau + 2\pi)) \]

Periodic Attitude Motion with period \( n \)-times the orbital period \( T \)

\[ P^n(\phi_0, \phi'_0) = (\phi_0, \phi'_0) \]

This corresponds to a zero of the function

\[ F = (P^n - \text{id})(\phi_0, \phi'_0) = (0, 0) \]

Find a zero by Newton-Raphson procedure

\[ (\phi_1, \phi'_1) = (\phi_0, \phi'_0) - (\nabla F)^{-1} F(\phi_0, \phi'_0) \]

Iterates stop when

\[ (\phi_{i+1}, \phi'_{i+1}) \sim (\phi_i, \phi'_i) \]
Periodic solutions with period $T$ (i.e. $2\pi$)

Tethered Satellite ($\alpha = 3$)
Eccentricity $e=0.2$
Periodic solutions with period $2T$ (i.e. $4\pi$)

Tethered Satellite ($\alpha = 3$)
Eccentricity $e=0.2$

Note that the spacecraft is pointing the Earth for long time at the apogee region.
Periodic solutions period 2 T: stability under J2 effect

Orbital Periods
(Stable) Periodic solutions by Cell Mapping Approach

Region filled by 1600 cells

Output 1: periodic solution
Output 2: Asymptotic to periodic solution
Output 3: lost cells

Starting Cell

$1 < \phi < 2$

$-45^o < \phi < +45^o$
\[ \alpha = 0.1 \]

\[ e = 0.3 \]

\[ -45^\circ < \phi < +45^\circ \]

*Increasing period*
Periodic Solutions

$n=58$ Stable Periodic Solution

$n=4$ Unstable Periodic Solution

$\alpha = 0.1 \quad e = 0.2$
Chaos

C1: sensibility to initial conditions
C2: There is an infinite number of periodic solutions
C3: There are some solutions approaching any region of the phase space (ergodicity)

Splitting of the separatrix due to perturbation \((e \neq 0)\) produces unpredictable attitude dynamics with many changes from rotation to oscillation state and vice versa.
Poincaré maps

\[ \alpha = 0.1 \]
\[ e = 0.1 \]
Rotations/Oscillations
Separatrix splitting

For moderate values of $e$, both regular and irregular motions are present in phase space. Chaotic behaviour occurs near the separatrix splitting.

Regions on the phase space trapped by separatrix splitting are mapped into regions of the same kind, with shifts between oscillation and rotation regimes.

Separatrix splitting was proved by computation of the Melnikov function.
No regular motion

For higher values of $e$ disorder appears everywhere in phase space

Iterates of Poincaré map of the point $(0,0)$, $e = 0.4$

Separatrix splitting transport mechanism can fill the phase space region. Also splitting of stable and unstable manifolds of periodic orbits may occur far from the separatrix.
Liapunov characteristic exponents

$X_0(t, x_0)$ solution of $\dot{x} = F(t, x)$ with i.c. $x(0) = x_0$

**STABILITY** $\quad \delta \dot{X} = \nabla F(X_0) \delta X$

$X_0$ equilibrium point

Time independent Linear System: Eigenvalues of the matrix

$X_0$ periodic solution

Linear System with periodic coefficients: Floquet exponents

$X_0$ generic solution

Dynamic around a trajectory of a generic dyn. syst.: Lyapunov exponents

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \| m_i(t) \|, \quad m_i \text{ eigenvalues of the Wronskian}$$
Chaos index

If one Lyapunov characteristic exponent of the (0,0) trajectory is positive, then the dynamics is chaotic.

Nevertheless periodic solutions exist.

\[ \alpha = 3 \]
\[ e = 0.4 \]
Is gravity gradient stabilization feasible?

P. Hughes “Spacecraft attitude dynamics” Wiley 1986

Flight experience of: APL, OV-10, Dodge, RAE, NRL-164, GEOS I, GEOS II

“It is clear that gravity gradient stabilization falls far short of sufficient for most modern applications. The best people in the business tried many configurations and many types of damping, yet their success was quite limited.

Nevertheless for certain special situations, particularly in near Earth orbits, can profitably be used when accurate pointing is not required.”
Gravity gradient stabilized Celestial bodies

Table 2. The values of parameters $e$ and $\mu$ for the resonance satellites in the Solar System

<table>
<thead>
<tr>
<th>Planet</th>
<th>Satellite</th>
<th>$e$</th>
<th>$\mu = \frac{\alpha}{3}$</th>
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</thead>
<tbody>
<tr>
<td>Earth</td>
<td>Moon</td>
<td>0.0549</td>
<td>0.00069</td>
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<tr>
<td>Mars</td>
<td>Phobos</td>
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<td>Deimos</td>
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<td>Io</td>
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</tr>
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TIGRISAT (2014)
THANKS

Vladimir Beletskii