On the Beletskii equation

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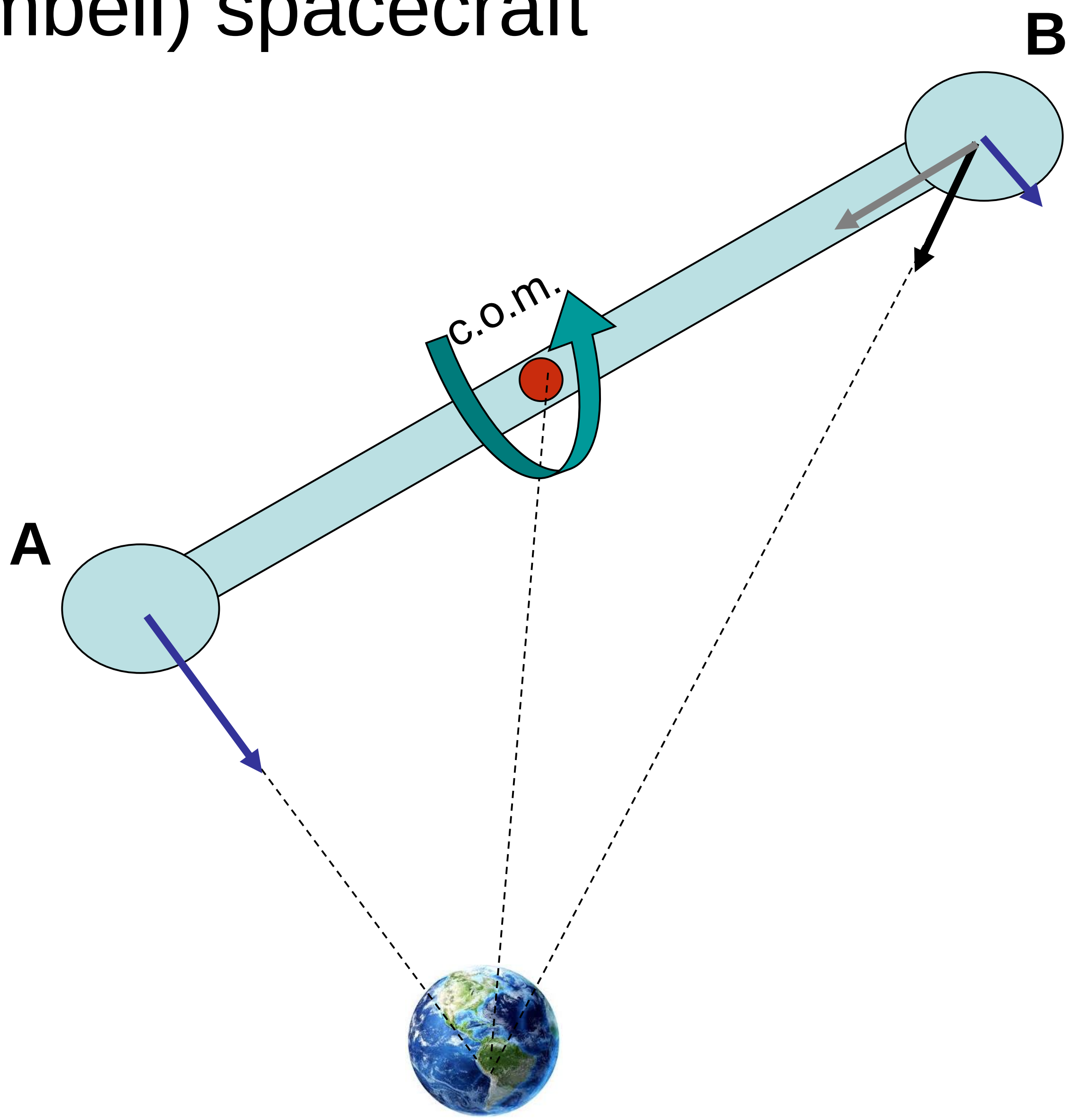
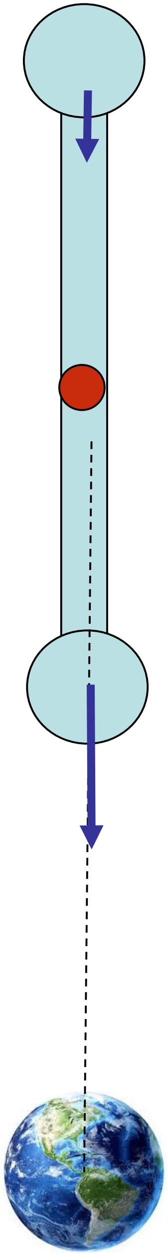
4th IAA Conference on University Satellite Missions and CubeSat Workshop

Rome, 4-7 December 2017

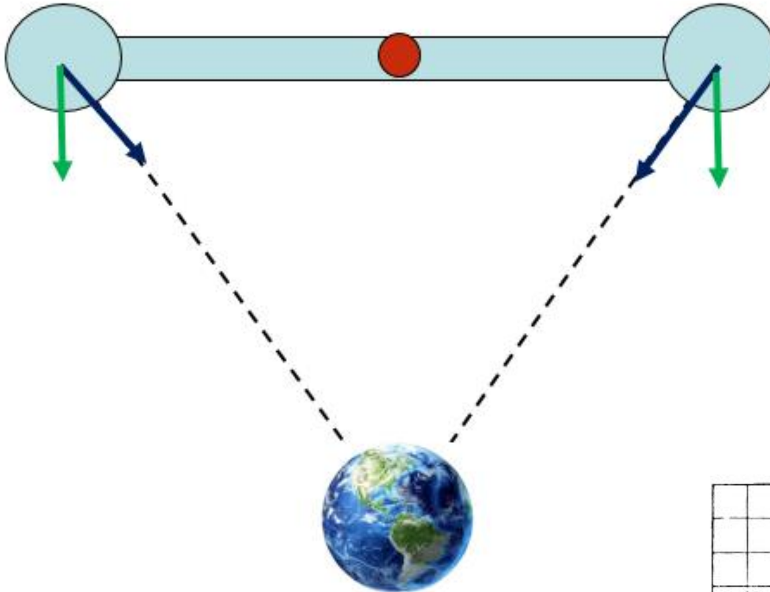
Gravity gradient effect on attitude motion of a (dumbell) spacecraft



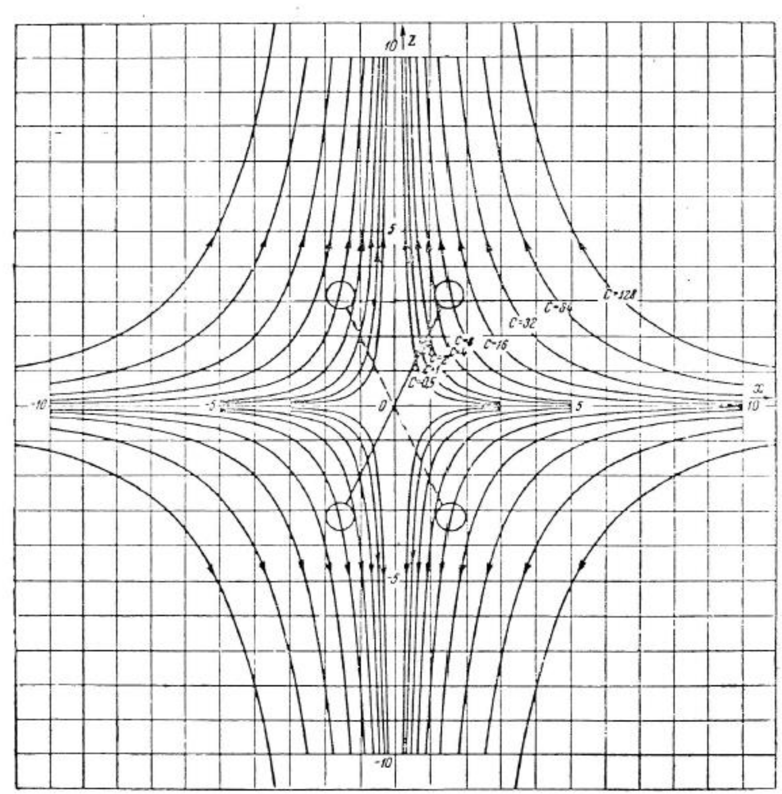
Equilibrium configurations



Stable Equilibrium



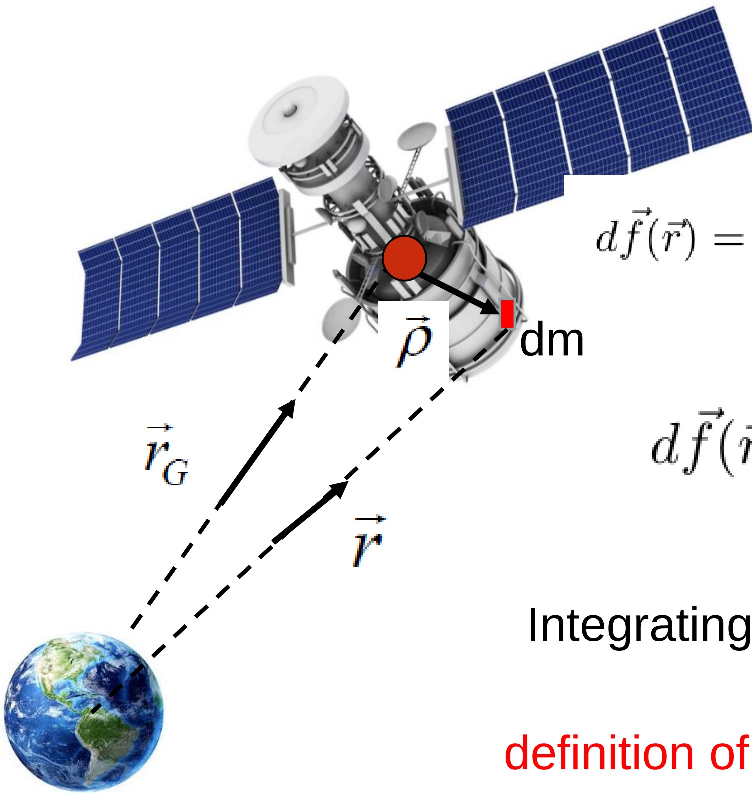
Unstable Equilibrium



Formulas

Gravitational torque wrt the c.o.m. G on a particle dm

$$d\vec{M} = \vec{\rho} \wedge d\vec{f}(\vec{r}) = \vec{\rho} \wedge \left(-\frac{\mu}{r^3}\vec{r} \right) dm$$



$$\vec{r} = \vec{r}_G + \vec{\rho}$$

$$d\vec{f}(\vec{r}) = -\frac{\mu\, dm}{r^3}\vec{r} = -\frac{\mu\, dm}{\|\vec{r}+\vec{r}_G\|^3}(\vec{r}+\vec{r}_G) \sim d\vec{f}(\vec{r}_G) + \nabla d f(\vec{r}_G)\,\vec{\rho}$$

$$d\vec{f}(\vec{r}) \sim -\frac{\mu}{r_G^3}\vec{r}_G\, dm + \frac{\mu}{r_G^3}\left(3\,\hat{r}_G\hat{r}_G - \mathbf{u}\right)\vec{\rho}\, dm$$

Integrating over all the mass of the satellite

definition of c.o.m.

vector product of $\vec{\rho}$ by itself

$$\vec{M} = \int_B d\vec{M} = \int_B \vec{\rho}\, \cancel{dm \wedge \left(-\frac{\mu}{r_G^3}\vec{r}_G\right)} + \frac{\mu}{r_G^3}\int_B \vec{\rho} \wedge \left(3\,\hat{r}_G\hat{r}_G - \cancel{\mathbf{u}}\right)\vec{\rho}\, dm$$

Gravity Gradient Torque and Equations of attitude motion

$$\vec{M} = 3\frac{\mu}{r_G^3} \int_B \vec{\rho} \wedge \hat{r}_G \hat{r}_G \; \vec{\rho} \, dm = -3\frac{\mu}{r_G^3} \int_B \hat{r}_G \wedge \vec{\rho}\vec{\rho} \, \vec{r}_G \, dm$$

Adding a «zero» we have

$$\vec{M} = 3\frac{\mu}{r_G^3} \, \vec{r}_G \wedge \int_B \left(\rho^2 \, \mathbf{u} - \vec{\rho}\vec{\rho}\right) dm \;\; \vec{r}_G$$

That is

$$\vec{M} = 3\frac{\mu}{r_G^3} \, \vec{r}_G \wedge \mathbf{I} \, \vec{r}_G$$

Gravitational Torque
in body frame

$$\mathbf{I}\dot{\vec{\Omega}} + \vec{\Omega} \wedge \mathbf{I}\vec{\Omega} = 3\frac{\mu}{r_G^3} \, \vec{r}_G \wedge \mathbf{I} \, \vec{r}_G$$

Body components and planar case

$$\vec{r}_G = r_x\,\hat{x}_B + r_y\,\hat{y}_B + r_z\,\hat{z}_B$$

$$\vec{\Omega} = \Omega_1\,\hat{x}_B + \Omega_2\,\hat{y}_B + \Omega_3\,\hat{z}_B$$

$$\dot{\Omega}_1 = -\frac{I_3 - I_2}{I_1}\Omega_3\,\Omega_2 + 3\frac{\mu}{r^3}\frac{I_3 - I_2}{I_1}r_y\,r_z$$

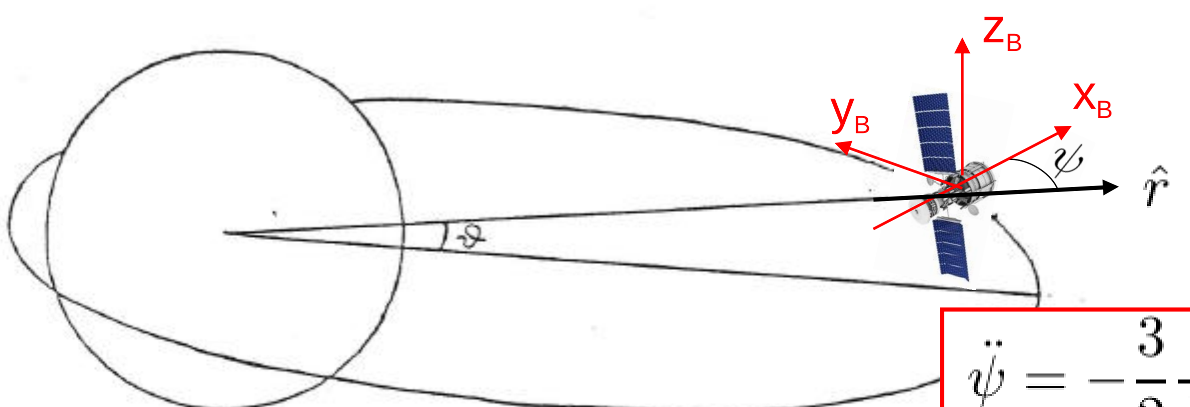
$$\dot{\Omega}_2 = -\frac{I_1 - I_3}{I_2}\Omega_3\,\Omega_1 + 3\frac{\mu}{r^3}\frac{I_1 - I_3}{I_1}r_x\,r_z$$

$$\dot{\Omega}_3 = -\frac{I_2 - I_1}{I_3}\Omega_2\,\Omega_1 + 3\frac{\mu}{r^3}\frac{I_2 - I_1}{I_3}r_x\,r_y$$

Planar Case $\Omega_1 = \Omega_2 = r_z = 0$

$$\dot{\Omega}_3 = 3\frac{\mu}{r^3}\frac{I_2 - I_1}{I_3}r_x\,r_y$$

$$\boxed{\ddot{\psi} + \ddot{\theta} = -3\frac{\mu}{r^3}\frac{I_2 - I_1}{I_3}\cos\psi\,\sin\psi}$$



e = 0

$$\boxed{\ddot{\psi} = -\frac{3}{2}\frac{\mu}{R^3}\frac{I_2 - I_1}{I_3}\sin 2\psi}$$

Consider elliptic orbits

$$\boxed{\ddot{\psi} + \ddot{\theta} = -3\,\frac{\mu}{r^3}\,\frac{I_2 - I_1}{I_3}\,\cos\psi\,\sin\psi}$$

$$r = \frac{h^2/\mu}{1 + e\,\cos\theta}$$

$$(*)' = \frac{d\,*}{d\,\theta}$$

$$\dot{\theta} = \frac{\mu^2}{h^3}(1 + e\cos\theta)$$

$$\ddot{\theta} = 2(\frac{\mu^2}{h^3})^2\,(1 + e\cos\theta)^3\,(-e\,\sin\theta)$$

$$\boxed{(1 + e\,\cos\theta)\,\phi'' - 2e\,\sin\theta\,\phi' + \alpha\,\sin\phi = 4e\,\sin\theta}$$

BELETSKII EQUATION

$$\phi = 2\,\psi$$

$$\alpha = 3\,\frac{I_2 - I_1}{I_3} \quad \in [0, 3]$$

Artificial Earth Sat. vol 3 (1959)

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL TRANSLATION F-10

THE LIBRATION OF A SATELLITE*

By V. V. Beletskiy

In this paper the methods of Liapunov-Chetayev are applied to find conditions for the existence and stability of the relative equilibrium of a material body in orbit in a Newtonian central-force field and to consider the oscillations of the body around the position of relative equilibrium. The problem considered is an idealized motion, which actually occurs in the solar system (the motion of the moon relative to the earth and the possible motions of artificial earth satellites).

Periodic solutions

Chaotic dynamics

Generalized equations

Periodic Solutions

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Generalized equations

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Periodic solutions of small amplitude and small eccentricity: Beletskii transformation

Small amplitude

$$(1 + e\ \cos\theta)\phi'' - 2\, e \sin\theta\, \phi' + 3\alpha\, \phi = 4e \sin\theta$$

BELETSKII TRANSFORMATION

$$z = (1 + e\ \cos\theta)\frac{\phi}{2}$$

$$z'' + \frac{\alpha + e\cos\theta}{1 + e\cos\theta}\, z = 2e\sin\theta$$

Small eccentricity

$$z'' + \alpha\, z = 2\, e \sin\theta + (\alpha - 1)(e\cos\theta - e^2\cos^2\theta)z + \ldots$$

For $e = 0$ the solution is:

$$\begin{cases} z = D\cos\tau \\ D' = 0 \\ \tau' = \sqrt(\alpha) \end{cases}$$

Periodic solutions of small amplitude and small eccentricity: solution

For $e << 1$ search a solution:

$$z = D\cos\tau + e\,u_1(D,\tau,\theta) + e^2\,u_1(D,\tau,\theta) + \ldots$$

with

$$D' = e\,A_1(D) + e^2\,A_2(D) + \ldots$$

$$\tau' = \sqrt(\alpha) + e\,B_1(D) + e^2\,B_2(D) + \ldots$$

Result

$$z(\theta) = D\cos(\sqrt{\alpha}\,\theta + e^2 B_2\theta + tau_0) + e\,u_1 + e^2\,u_2$$

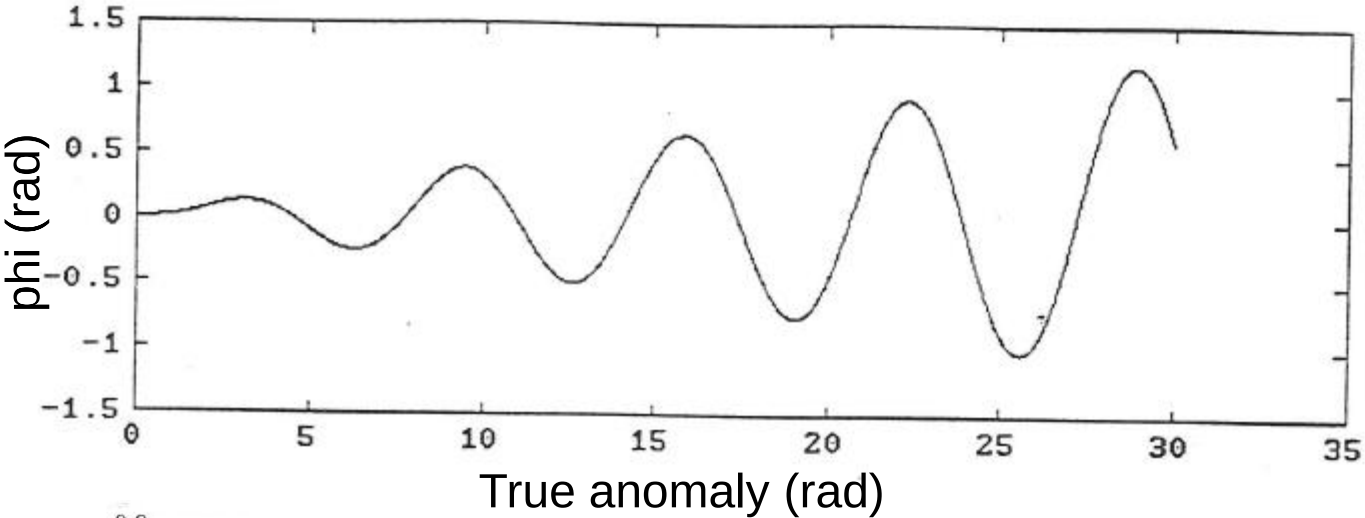
resonances

$$u_1 = \frac{2\sin\theta}{\alpha - 1} + \frac{D(\alpha-1)}{2}\cos((\Gamma-1)\theta + \tau_0) - \frac{2\sqrt{\alpha}-1}{4\alpha-1}\cos((\Gamma+1)\theta + \tau_0)$$

$$u_2 = \frac{\sin(2\theta)}{\alpha-4} + \frac{\alpha}{16}\,\frac{\sqrt{\alpha}(\sqrt{\alpha}+1)(\sqrt{\alpha}(\sqrt{\alpha}-2)}{2\sqrt{\alpha}+1}\cos((\Gamma-2)\theta+\tau_0) + \frac{\sqrt{\alpha}(\sqrt{\alpha}-1)(\sqrt{\alpha}(\sqrt{\alpha}+2}{2\sqrt{\alpha}+1}\cos((\Gamma+2)\theta+\tau_0)$$

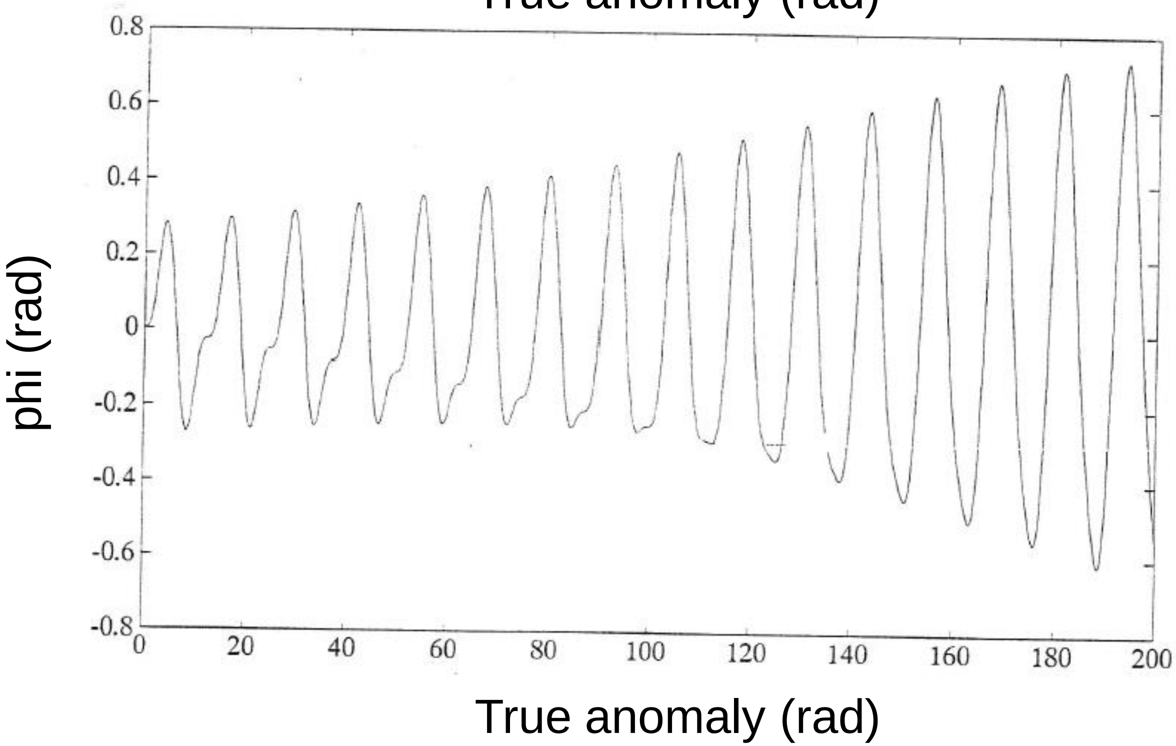
$$\Gamma = \sqrt{\alpha} + e^2\,B_2 \quad , \quad B_2 = \frac{3}{4}\,\frac{\sqrt{\alpha}(\alpha-1)}{4\,\alpha-1}$$

Numerical solutions: resonances



Initial state = (0,0)

alpha = 1



Initial state = (0,0)

alpha = 1/4

Periodic Solutions by Newton-Raphson on the Poincarè map

$$\mathbf{P} \; : \; (\phi(\tau), \phi'(\tau)) \; \longrightarrow \; (\phi(\tau + 2\pi), \phi'(\tau + 2\pi))$$

Periodic Attitude Motion with period n-times the orbital period T

$$P^n(\phi_0, \phi_0') = (\phi_0, \phi_0')$$

This corresponds to a zero of the function

$$F = (P^n - \mathbf{id})(\phi_0, {\phi'}_0) = (0, 0)$$

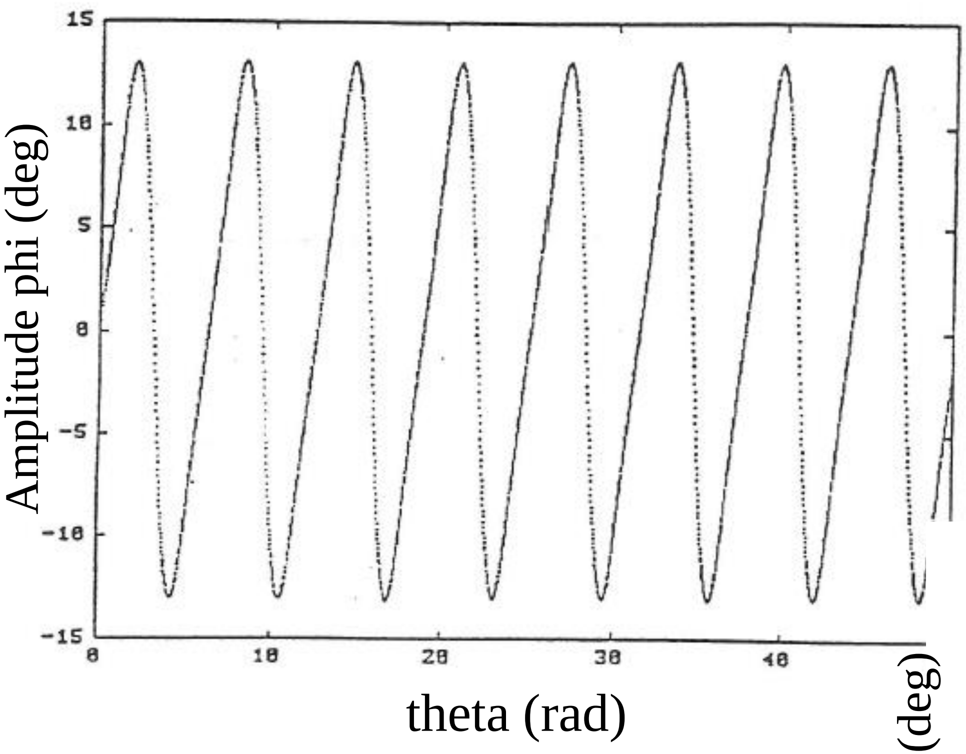
Find a zero by Newton-Raphson procedure

$$(\phi_1, \phi_1') = (\phi_0, \phi_0') - (\nabla F)^{-1} F(\phi_0, \phi_0')$$

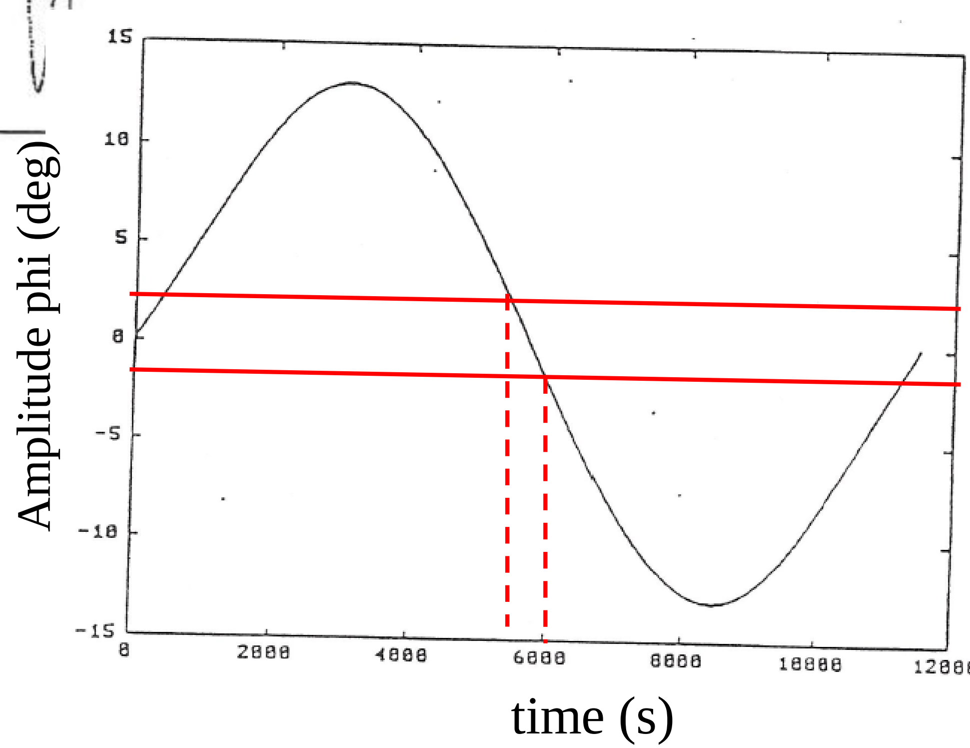
Iterates stop when

$$(\phi_{i+1}, \phi_{i+1}') \sim (\phi_i, \phi_i')$$

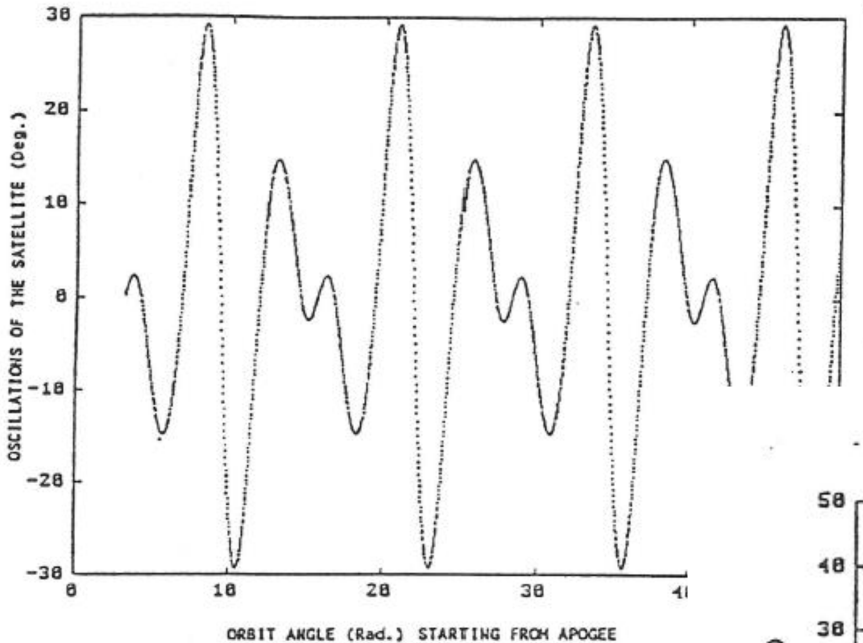
Periodic solutions with period T (i.e. 2*pi)



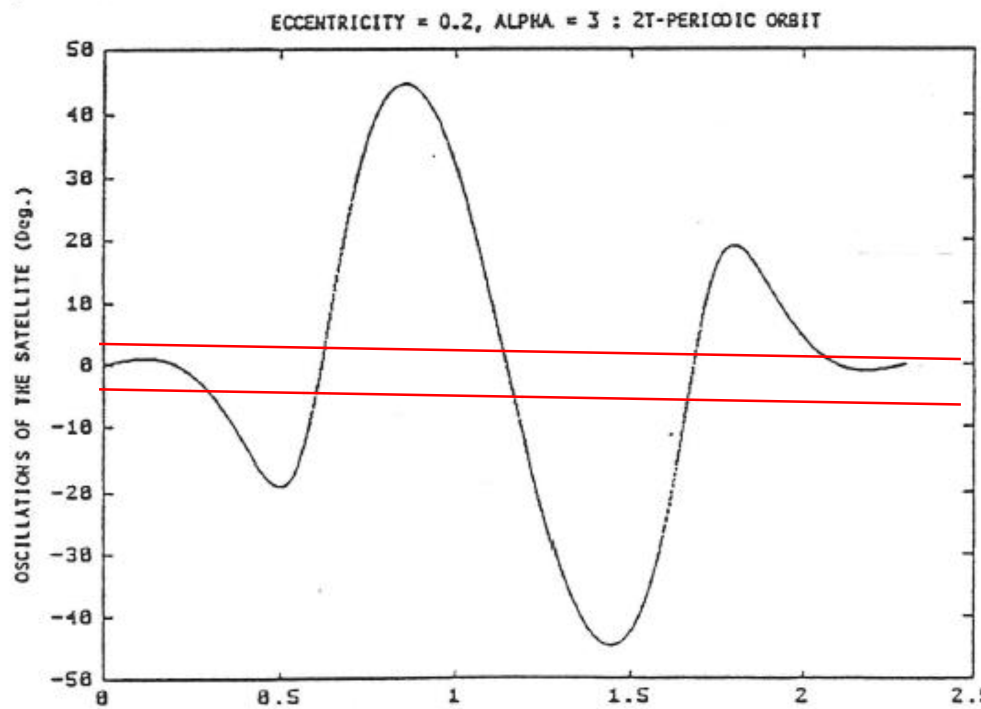
Tethered Satellite (alfa = 3)
Eccentricity e=0.2



Periodic solutions with period 2 T (i.e. 4*pi)



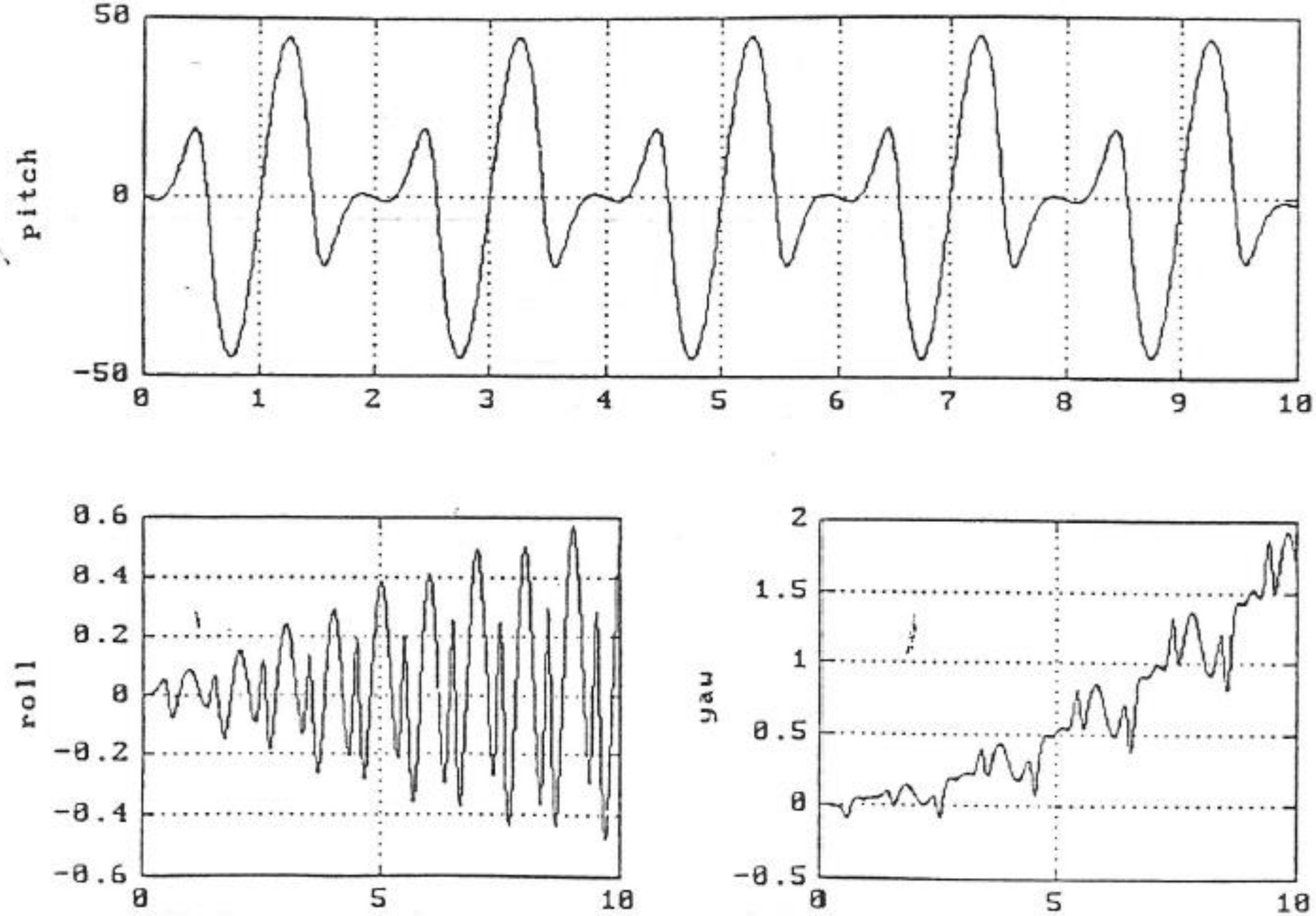
Tethered Satellite (alfa = 3)
Eccentricity e=0.2



Orbital Periods

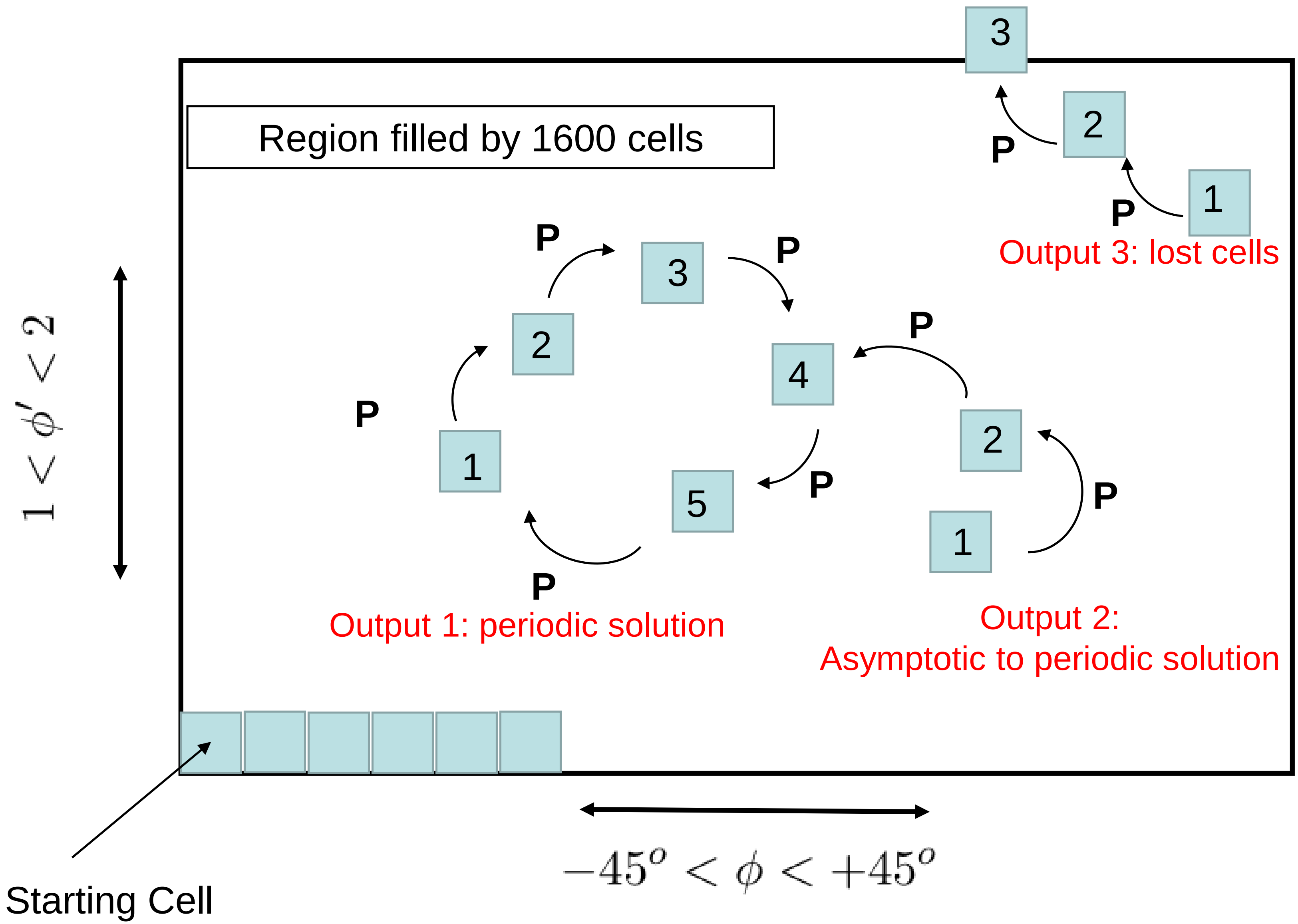
Note that the spacecraft is pointing the Earth for long time at the apogee region

Periodic solutions period 2 T: stability under J2 effect



Orbital Periods

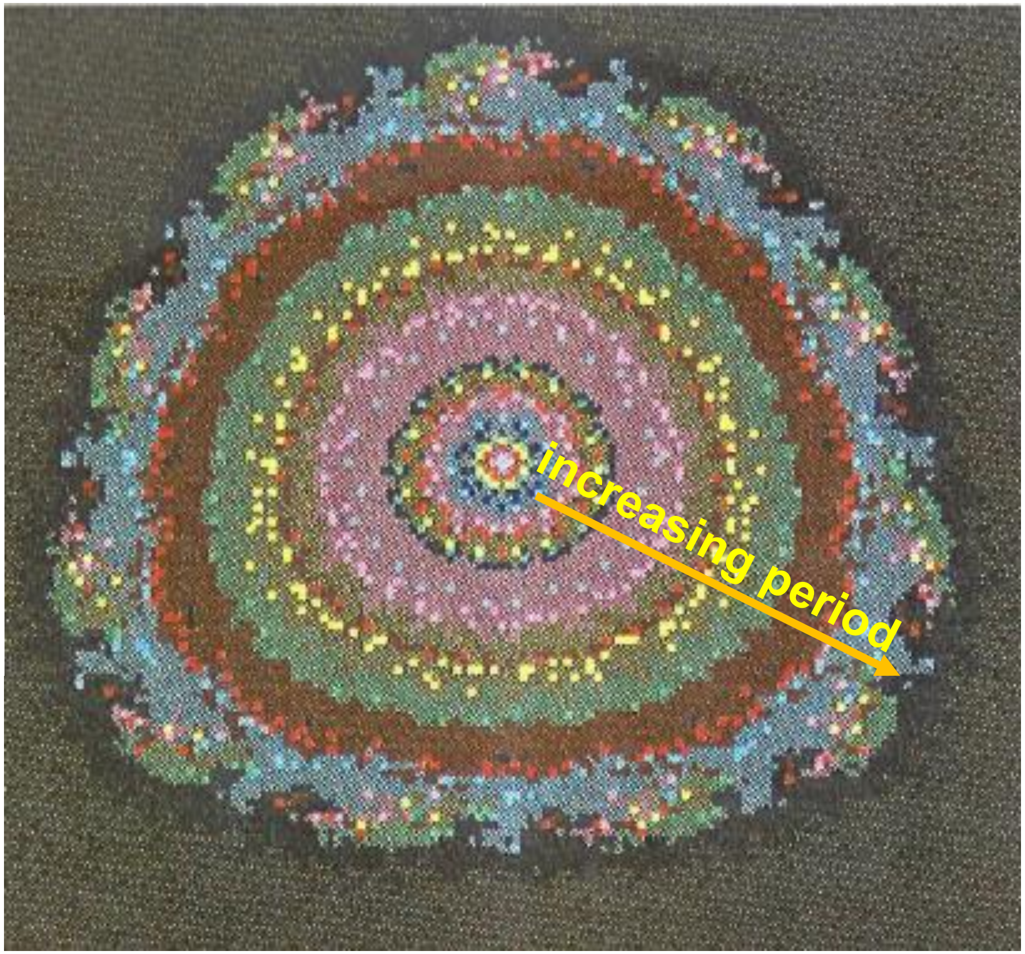
(Stable) Periodic solutions by Cell Mapping Approach



Cell Mapping Output

$\alpha = 0.1$

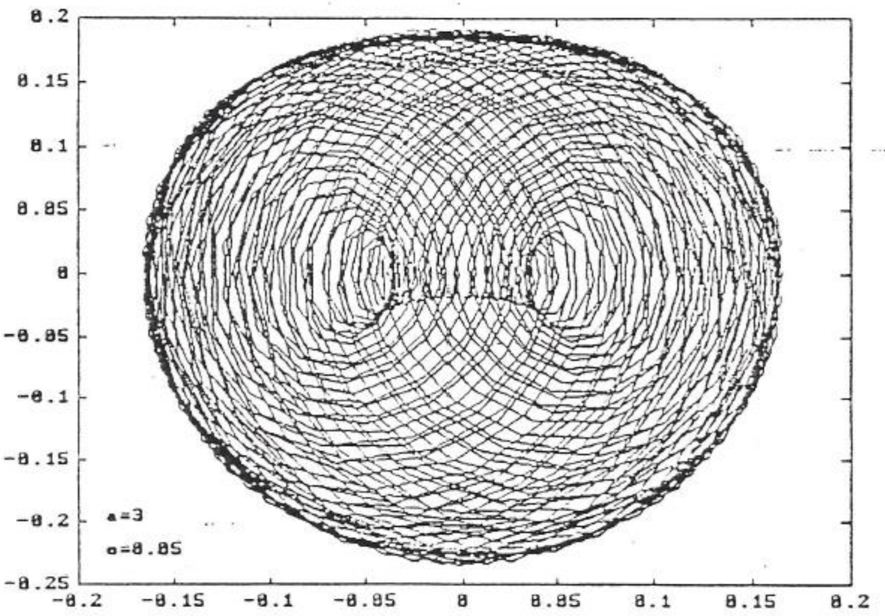
$e = 0.3$



$1 < \phi' < 2$

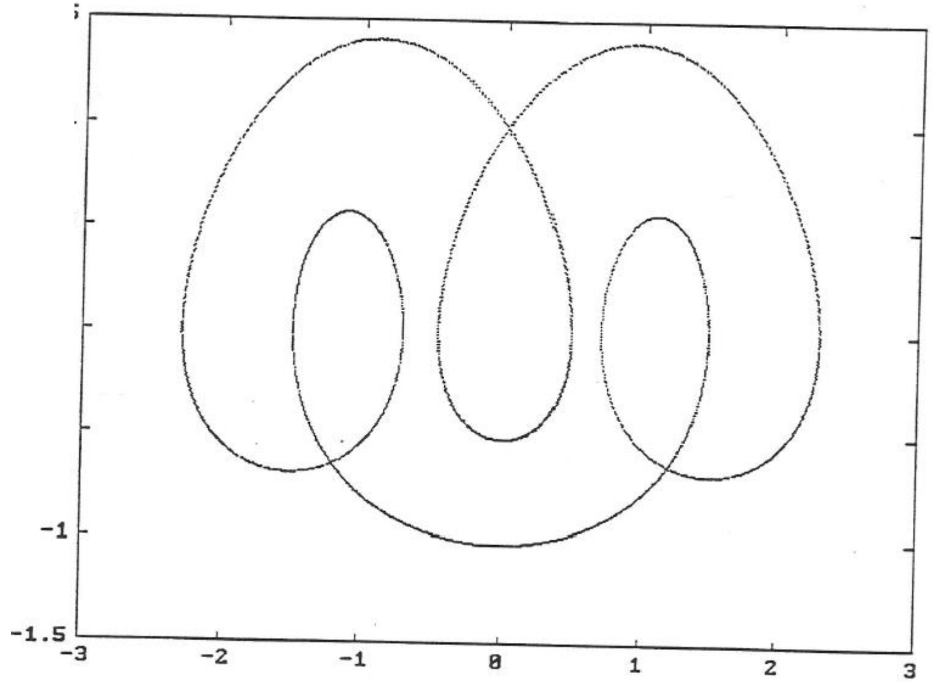
$-45^o < \phi < +45^o$

Periodic Solutions



n=58 Stable Periodic Solution

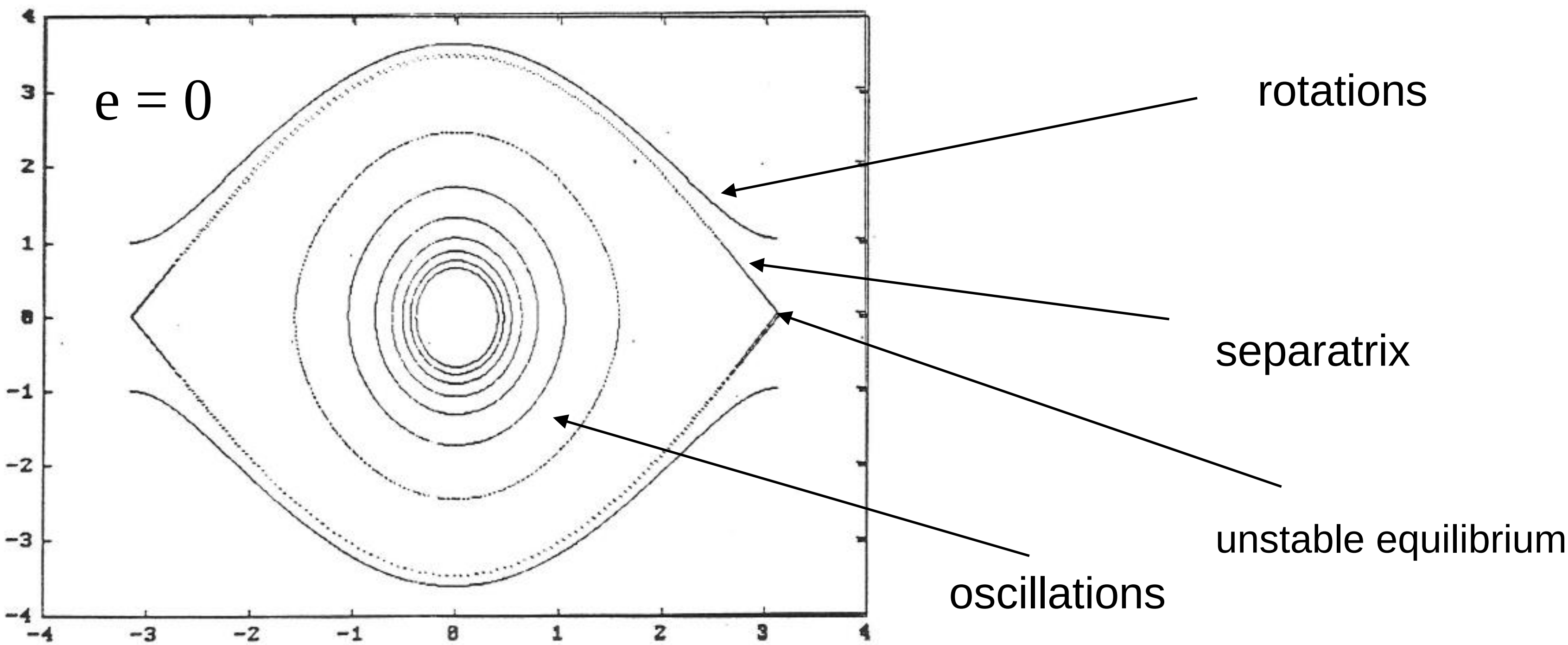
$\alpha = 0.1 \qquad e = 0.2$



n=4 Unstable Periodic Solution

Chaos

C1 : sensibility to initial conditions
C2 : There is an infinite number of periodic solutions
C3 : There are some solutions approaching any region of the phase space (ergodicity)

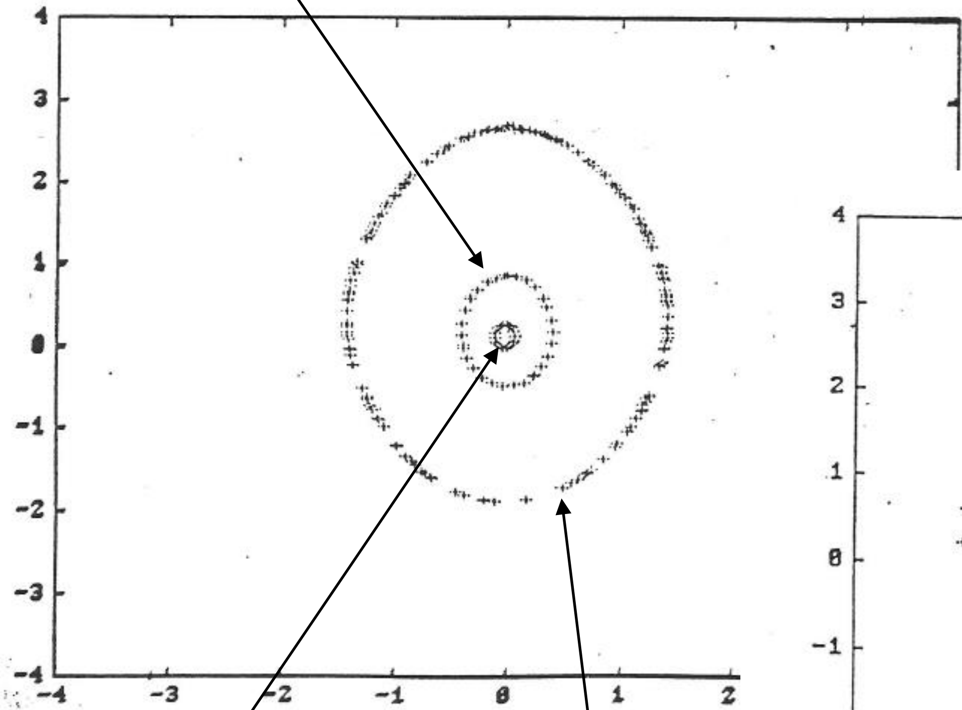


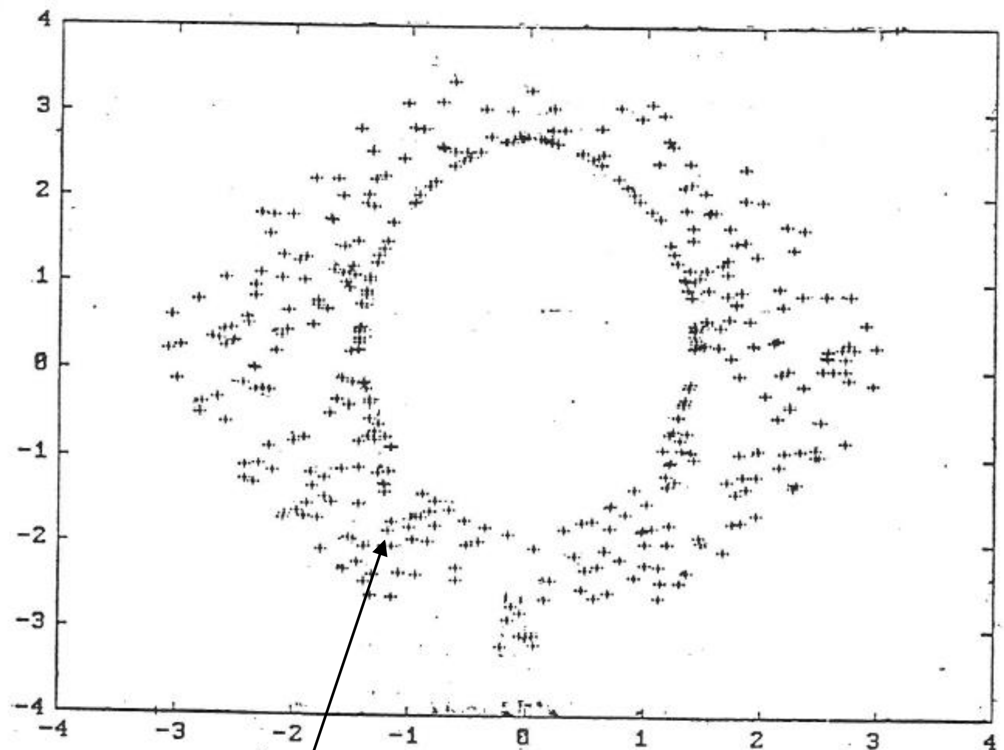
Splitting of the separatrix due to perturbation $(e \neq 0)$ produces unpredictable attitude dynamics with many changes from rotation to oscillation state and viceversa

Poincaré maps

Poincaré map of (0.4,0)

$$\alpha = 0.1$$
$$e = 0.1$$



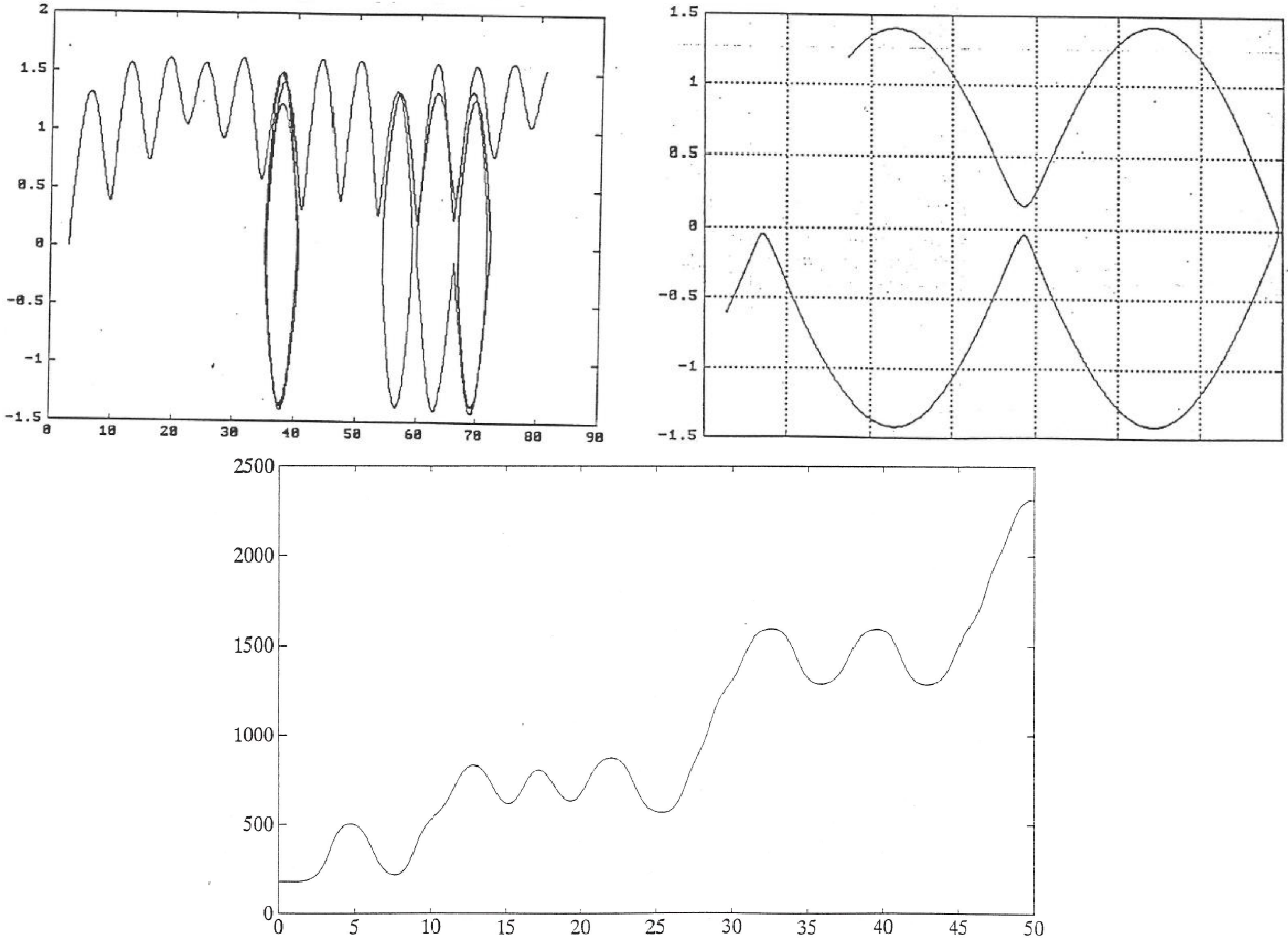


Poincaré map of (0,0)

Poincaré map of (1.41,0)

Poincaré map of (1.42,0)

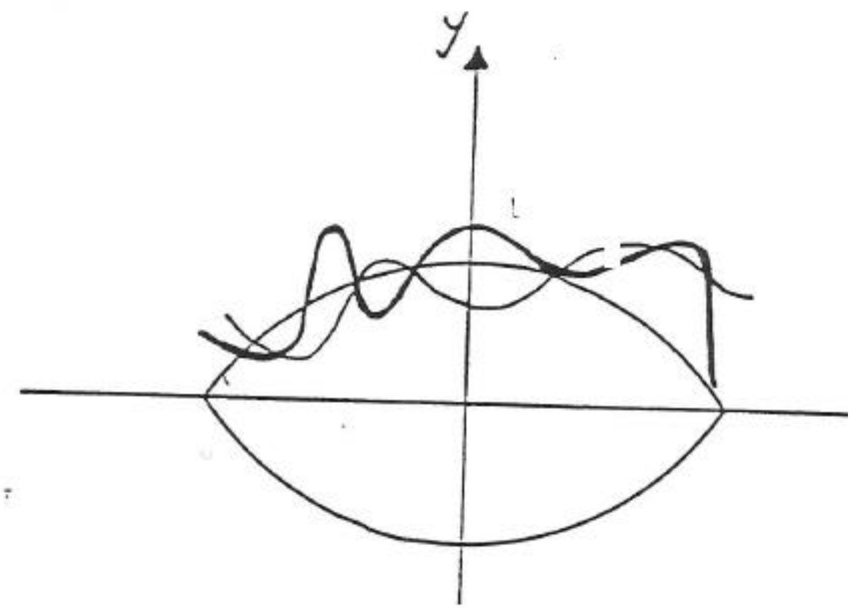
Rotations/Oscillations

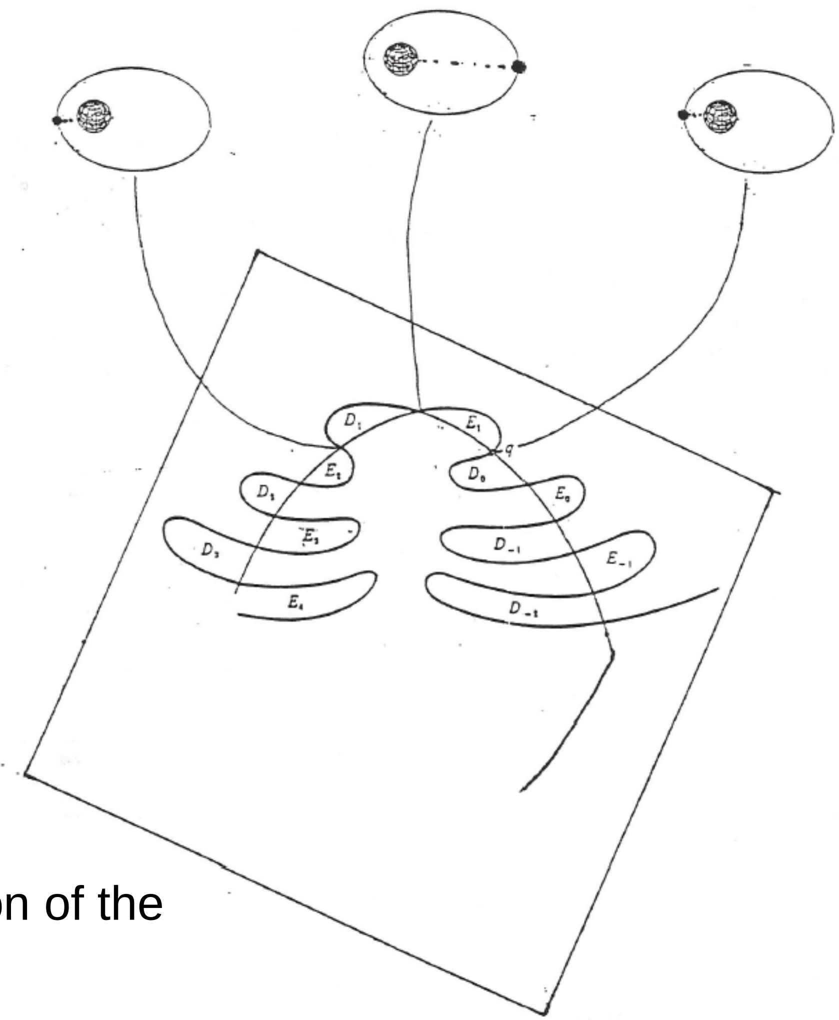


orbital periods

Separatrix splitting

For moderate values of *e* , both regular and irregular motions are present in phase space. Chaotic behaviour occurs near the separatrix splitting



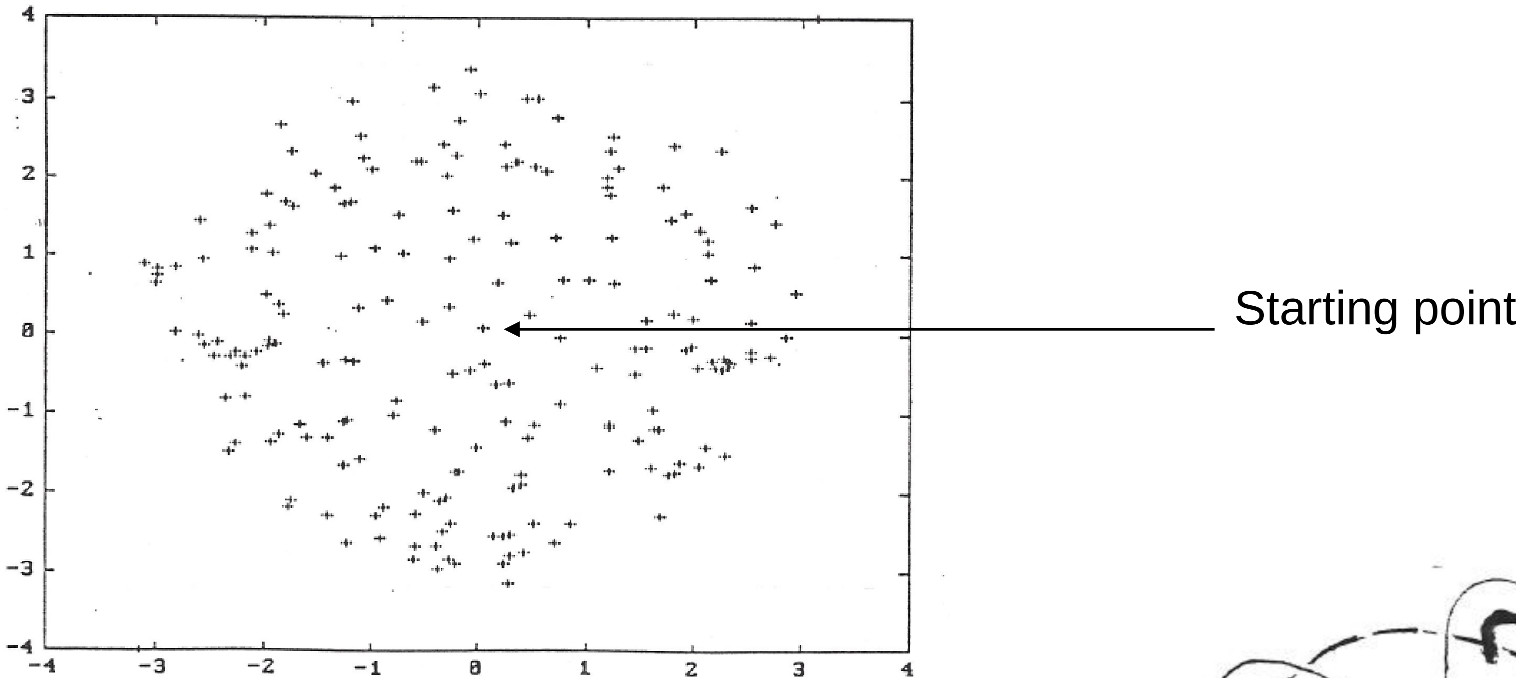


Regions on the phase space trapped by separatrix splitting are mapped into regions of the same kind, with shifts between oscillation and rotation regimes

Separatrix splitting was proved by computation of the Melnikov function

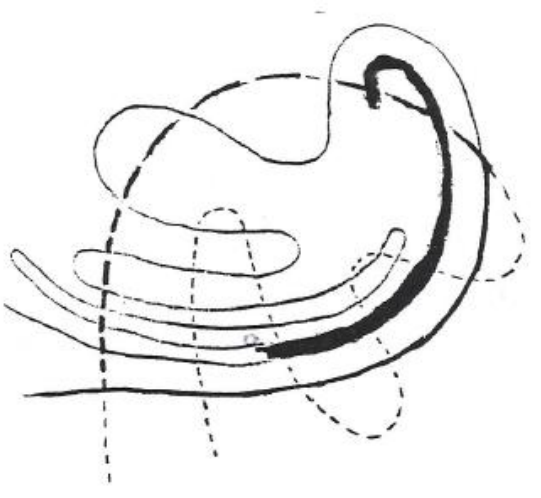
No regular motion

For higher values of *e* disorder appears everywhere in phase space



Iterates of Poincaré map of the point (0,0) , e = 0.4

Separatrix splitting tranport mechanism can fill the phase space region. Also splitting of stable and unstable manifolds of periodic orbits may occur far from the separatrix



Liapunov characteristic exponents

$$X_0(t, x_0) \;\; solution \;\; of \; \dot{x} = F(t, x) \; with \; i.c. \; x(0) = x_0$$

STABILITY $\longrightarrow$ $\delta \dot{X} = \nabla F(X_0) \, \delta X$

$$X_0 \;\; equilibrium \; point$$

Time independent Linear System : Eigenvalues of the matrix

$$X_0 \;\; periodic \; solution$$

Linear System with periodic coefficients: Floquet exponents

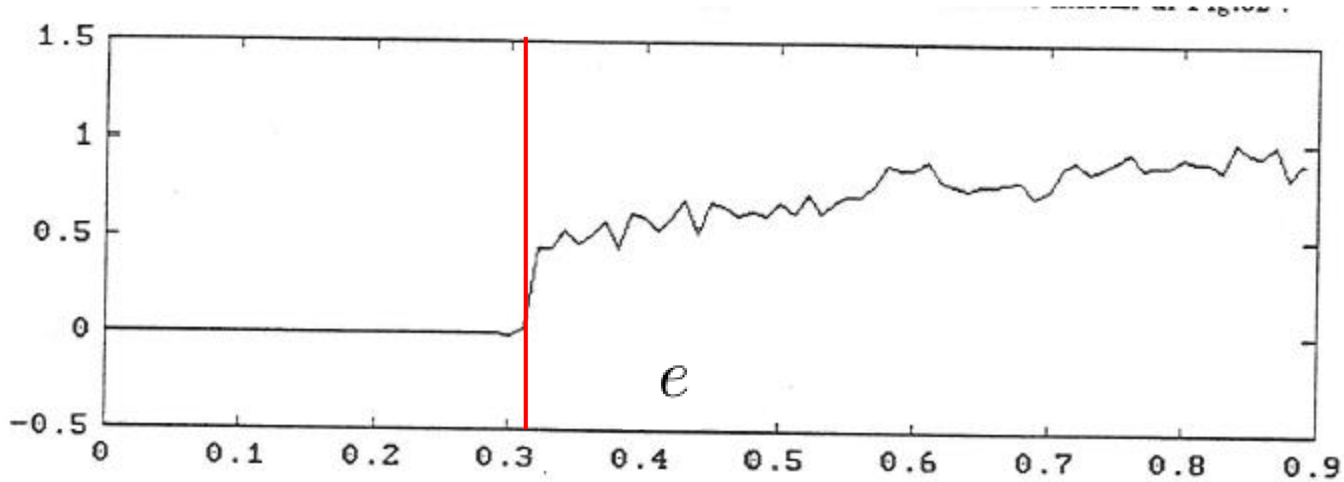
$$X_0 \;\; generic \; solution$$

Dynamic around a trajectory of a generic dyn. syst.: Lyapunov exponents

$$\lambda_i = lim_{t \longrightarrow \infty} \frac{1}{t} log \| m_i(t) \| \;\; , \;\; m_i \;\; eigenvalues \; of \; the \; Wronskian$$

Chaos index

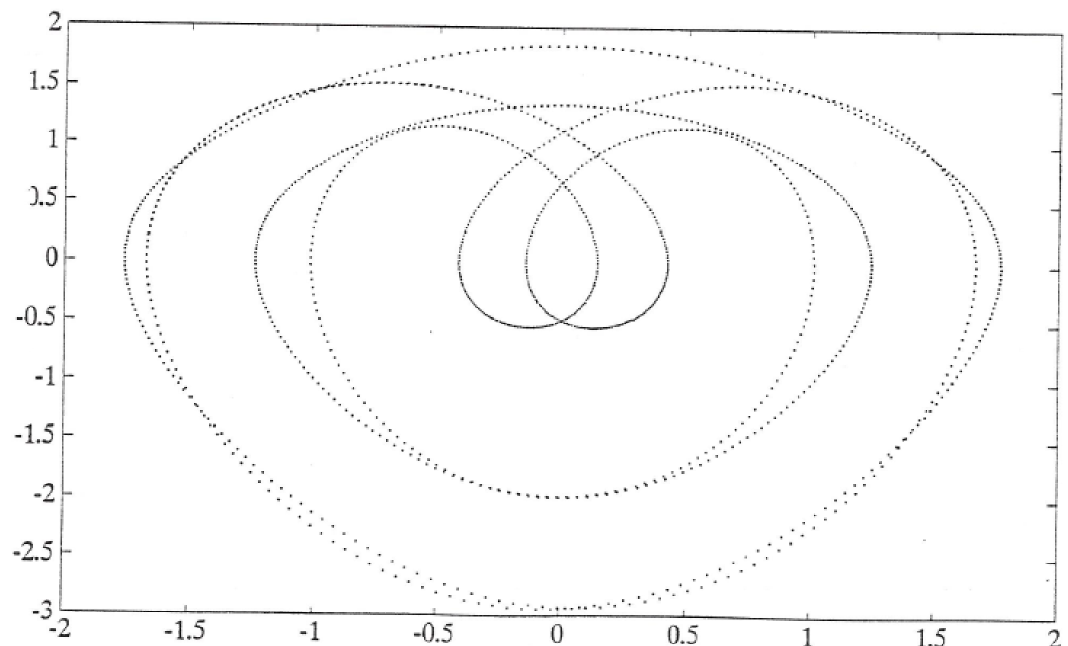
If one Lyapunov characteristic exponent of the (0,0) trajectory is positive, then the dynamics is chaotic



Neverthless periodic solutions exist

$$\alpha = 3$$

$$e = 0.4$$



Is gravity gradient stabilization feasible ?

P.Hughes “Spacecraft attitude dynamics” Wiley 1986

Flight experience of : APL, OV-10 , Dodge , RAE, NRL-164.
GEOS I, GEOS II

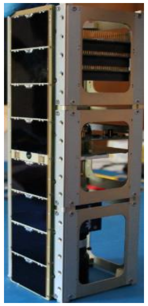
“It is clear that gravity gradient stabilization falls far short of sufficient for most modern applications. The best people in the business tried many configurations and many types of damping , yet their success was quite limited.

Neverthless for certain special situations, particularly in near Earth orbits, can profitably be used when accurate pointing is not required.”

Gravity gradient stabilized Celestial bodies

Table 2. The values of parameters e and μ for the resonance satellites in the Solar System

Planet	Satellite	e	$\mu = \frac{\alpha}{3}$
Earth	Moon	0.0549	0.00069
Mars	Phobos	0.0150	0.684
	Deimos	0.00052	0.612
Jupiter	Io	0.0041	0.021
	Europa	0.0094	0.0055
	Ganymede	0.0014	0.002
	Callisto	0.0074	0.00055
Saturn	Enceladus	0.00532	0.087
	Tethys	0.00021	0.05
	Dione	0.00171	0.02
	Rhea	0.00021	0.007
	Iapetus	0.02882	0.0001
Neptune	Triton	0.0005	0.00066
Sun	Mercury	0.20563	0.0003



TIGRISAT (2014)

THANKS

Vladimir Beletskii